

Dynamical growth of the hadron bubbles during the quark-hadron phase transition

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The rate of dynamical growth of the hadron bubbles in a supercooled baryon free quark-gluon plasma, is evaluated by solving the equations of relativistic fluid dynamics in all regions. For a non-viscous plasma, this dynamical growth rate is found to depend only on the range of correlation ξ of order parameter fluctuation, and the radius R of the critical hadron bubble, the two length scales relevant for the description of the critical phenomena. Further, it is shown that the dynamical prefactor acquires an additive component when the medium becomes viscous. Interestingly, under certain reasonable assumption for the velocity of the sound in the medium around the saddle configuration, the viscous and the non-viscous parts of the prefactor are found to be similar to the results obtained by Csernai-Kapusta and Ruggeri-Friedman (for the case of zero viscosity) respectively.

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I. INTRODUCTION

The phenomena of phase transition has attracted many researchers from diverse areas due to many interesting and common features that occur near the transition point. Recently, a considerable amount of attention is being paid to the study of relativistic heavy ion collisions where a phase transition is expected from the normal nuclear matter to a deconfined state of quarks and gluons [1]. The quark gluon plasma (QGP), if formed, would expand hydrodynamically and would cool down until it reaches a critical temperature T_c where a phase transition to hadron phase begins. Although the order of such a phase transition remains an unsettled issue, a considerable amount of work has been carried out to understand the dynamics assuming it to be of first order and also assuming that the homogeneous nucleation is applicable [2–4]. In the ideal Maxwell construction, the temperature of the plasma remains fixed at T_c during the phase transition until the hadronization gets completed. However, if the hadronization proceeds through nucleation, it will not begin at $T = T_c$ due to the large nucleation barrier. The nucleation of the hadron bubbles can begin only from a supercooled metastable state. If the amount of supercooling is small, the nucleation rate [5] is computed from $I = A \exp(-\Delta F/T)$ which gives the probability per unit time per unit volume to nucleate a region of the stable phase (the hadron phase) within the metastable phase (the QGP phase). Here ΔF is the minimum energy needed to create a critical bubble and the prefactor A is the product of statistical and dynamical factors. The statistical factor Ω_0 is a measure of both the available phase space as the system goes over the saddle and of the statistical fluctuations at the saddle relative to the equilibrium states. The dynamical prefactor κ gives the exponential growth rate of the bubble or droplet sitting on the saddle.

In an earlier work, Langer and Turski [6] derived the dynamical growth rate (κ) of the liquid droplet based on a non-relativistic formalism. Subsequently, κ was derived both by Turski-Langer [7] and Kawasaki [8] for a liquid-

gas phase transition near the critical point, to be

$$\kappa = \frac{2\lambda\sigma T}{l^2 n_l^2 R^3}, \quad (1)$$

which involves the thermal conductivity λ , the surface free energy σ , the latent heat per molecule l and the density of the molecules in the liquid phase n_l . The interesting physics in this expression is the thermal conductivity which appears as an essential ingredient for the transportation of the latent heat away from the surface region so that the droplet can grow. For a relativistic system like a baryon free quark gluon plasma which has no net conserved charge, the thermal conductivity vanishes. Hence, the above formula obviously can not be applied to such systems. Therefore, Csernai and Kapusta re-derived κ for a baryon free plasma using earlier formalism of Langer-Turski [6], but extending their work to the relativistic domain [9]. In the work of Langer and Turski [6], κ was derived by solving a set of linearized hydrodynamic equations in the liquid, vapor and the interfacial regions. However, in the relativistic formalism, Csernai-Kapusta mostly concentrated in the interfacial region. Their primary motivation was to know the velocity profile in the surface region which was then used to estimate the energy flow across the surface. Then they used the condition, that the energy flux which is to be transported outwards should be balanced by the viscous heat dissipation as follows

$$\Delta\omega \frac{dR}{dt} = -(4\eta/3 + \zeta) v dv/dr, \quad (2)$$

where R is the radius of the hadron bubble and $v(r)$ is the flow velocity just outside the surface of the bubble. Accordingly, they obtained an expression for κ given by

$$\kappa = \frac{4\sigma(\frac{4}{3}\eta + \zeta)}{(\Delta\omega)^2 R^3}, \quad (3)$$

where η and ζ are the shear and bulk viscosity coefficients respectively and $\Delta\omega$ is the difference in the enthalpy densities of the plasma and the hadronic phases, $\omega = e + p$.

The above expression implies that energy flow $\omega \mathbf{v}$ is provided by the viscous effects. There will be no bubble growth in the case of an ideal plasma with zero viscosity. Thus, the viscosity plays the same role as thermal conductivity in case of a relativistic fluid like the quark gluon plasma with zero baryon density. This approach has been extended by Venugopalan and Vischer [10] for the case of baryon rich QGP where both viscous damping and thermal dissipation are significant. On the contrary, Ruggeri and Friedman (RF) [11] argued that the energy flow does not vanish in absence of any heat conduction or viscous damping. Since the change of the energy density e in time is given in the low velocity limit by the conservation equation [9], $\partial e / \partial t = -\nabla \cdot (\omega \mathbf{v})$ which implies that the energy flow $\propto \omega \mathbf{v}$ is always present. Therefore, following a different approach, Ruggeri and Friedman [11] derived an expression for κ which does not vanish in the absence of viscosity, given by

$$\kappa = \sqrt{\frac{2\sigma}{R^3} \frac{\omega_q}{(\Delta\omega)^2}}. \quad (4)$$

The viscous effects cause only small perturbations to the above equation. This result is in contradiction with the expression given in Eq. (3) according to which the hadron bubble will not grow in the limit of vanishing viscosity. The difference between Csernai-Kapusta (CK) and Ruggeri-Friedman (RF) results are due to the technical differences in the treatment of the pressure gradients and it needs further investigation. Motivated by this, we re-derive κ using Csernai-Kapusta formalism which is a relativistic generalization of Langer-Turski (LT) procedure [12]. However, unlike Csernai-Kapusta, we solve the linearized hydrodynamic equations in all regions namely the exterior quark region, the interior hadron region as well as the interfacial or the surface region. We found that in the limit of zero viscosity, our prefactor κ depends only on two scale parameters, the correlation length ξ and the critical radius of the hadron bubble R . We have also obtained the prefactor for a viscous medium where it can be written in a simple way as the sum of a viscous and a non-viscous terms. Interestingly, using certain assumption for the velocity of sound in the medium around the saddle configuration, the viscous and non-viscous components are found to be similar to the results as obtained by Csernai-Kapusta [Eq. (3)] and Ruggeri-Friedman [for zero viscosity, Eq. (4)] respectively.

The paper is organized as follows. We begin with a brief review of the Csernai-Kapusta and Turski-Langer formalism describing the energy-momentum conserving equations of motion in section II. In section III, we solve these equations to derive the dynamical prefactor. Finally, the numerical calculations and the conclusions are presented in sections IV and V respectively.

II. THE RELATIVISTIC HYDRODYNAMICS FOR BARYON FREE PLASMA

In the case of relativistic hydrodynamics, we consider the energy density $e(\mathbf{r}, t)$ and the flow velocity $\mathbf{v}(\mathbf{r}, t)$ of the fluid as two independent variables that describe the dynamics of the system. The equations of motion can be obtained from the local conservation laws:

$$\partial_\mu T^{\mu\nu} = \partial_\mu n^\mu = 0. \quad (5)$$

Here $T^{\mu\nu}$ is the energy momentum tensor and n^μ represents the baryon four vector. In the presence of viscosity, the energy-momentum tensor $T^{\mu\nu}$ and baryon four vector current n^μ can be decomposed into an ideal and a viscous part [13,14]

$$T^{\mu\nu} = [(e + p)u^\mu u^\nu - pg^{\mu\nu}] + \tau^{\mu\nu}, \quad (6)$$

$$n^\mu = nu^\mu + \nu^\mu. \quad (7)$$

Here e , p and n are the energy density, pressure and particle number density. The fluid four velocity is given by $u^\mu = \gamma(1, \mathbf{v})$ and $\tau^{\mu\nu}$ and ν^μ are the dissipative corrections. The form of the dissipative terms $\tau^{\mu\nu}$ and ν^μ depend on the definition of what constitutes the local rest frame of the fluid. The four velocity u^μ should be defined in such a way that in a proper frame of any given fluid element, the energy and the number densities are expressible in terms of other thermodynamic quantities by the same formulae, when dissipative processes are not present. It is also necessary to specify whether u^μ is the velocity of energy transport or particle transport. Accordingly, there exist two definitions for the rest frame; one due to Landau and other due to Eckart. In the Landau approach, u^μ is taken as the velocity of energy transport so that energy three flux T^{0i} vanishes in a comoving frame [13,14]. In the Eckart definition, u^μ is taken as the velocity of the particle transport and the particle three current, rather than the energy three flux vanishes in the fluid rest frame [14]. So in the Eckart definition of rest frame, the particle four vector can be written as $n^\mu = (n, \mathbf{0})$, whereas in the Landau definition of rest frame $n^\mu = (n, \nu)$. Therefore, the two frames are related by a Lorentz transformation with a boost velocity ν/n . It is found that due to ill defined boost velocity [15], the energy three flux in the Eckart frame (which involves heat conductivity λ) is not well defined as λ diverges in the limit of chemical potential $\mu \rightarrow 0$. On the other hand, in the Landau definition heat conduction enters as a correction to baryon flux. It was shown that inspite of the divergence of λ , the correction to the baryon flux ν^μ is finite [15]. Therefore, we will use the Landau definition for the subsequent study and also we will assume a baryon free plasma for simplicity. We can now write the equations of motion from the conservation law $\partial_\mu T^{\mu\nu} = 0$ using Landau definition [9,15]

$$\partial_t e = -\nabla \cdot (\omega \mathbf{v}) + \mathcal{O}(\mathbf{v}^2), \quad (8)$$

$$\begin{aligned} \partial_t(\omega \mathbf{v}) = & -\nabla \cdot (\omega \mathbf{v} \otimes \mathbf{v}) - \nabla p \\ & + \nabla \cdot \left(\left(\frac{4}{3} \eta + \zeta \right) \nabla \cdot \mathbf{v} \right) + \mathcal{O}(\mathbf{v}^2). \end{aligned} \quad (9)$$

Here $\omega = (e + p)$ and η and ζ are the shear and bulk viscosity coefficients respectively. We have also assumed the low speed limit where $\gamma \approx 1$. Although, the fluid velocity is small, the velocity of individual particles is large. Thus, the expressions for the energy density, pressure etc. are taken to be same as that in the relativistic case.

Following Ref. [9], Eq. (9) can also be written in terms of the Helmholtz free energy $f(e)$ and the usual gradient energy $\frac{1}{2}K(\nabla e)^2$ as

$$\begin{aligned} \partial_t(\omega \mathbf{v}) = & -\nabla \cdot (\omega \mathbf{v} \otimes \mathbf{v}) - \nabla p' \\ & + \nabla \cdot \left(\left(\frac{4}{3} \eta + \zeta \right) \nabla \cdot \mathbf{v} \right) + \mathcal{O}(\mathbf{v}^2), \end{aligned} \quad (10)$$

where

$$-\nabla p' = -K(\nabla^2 e) \nabla e + \frac{\partial f}{\partial e} \nabla e. \quad (11)$$

The constant K is related to the surface tension σ as

$$\sigma = K \int_{-\infty}^{\infty} dr \left(\frac{de}{dr} \right)^2. \quad (12)$$

It can be noted by comparing Eq. (10) with Eq. (9) that $-\nabla p'$ is not simply a pressure but a combination of ∇f and a force term $-K(\nabla^2 e) \nabla e$ which is related to the surface tension given by Eq. (12). The pressure inside the interface differs from that outside so there is necessarily a pressure gradient at the interface. The term given by Eq. (11) is needed to balance the differential pressure otherwise the Euler or the Navier Stokes equation would require a changing fluid velocity even in a stationary configuration, which is unphysical.

III. SOLUTION OF THE RELATIVISTIC HYDRODYNAMIC EQUATIONS

The above hydrodynamic equations [Eq. (9) or Eq. (10)] can be solved after linearizing around the saddle configuration. The saddle point corresponds to the stationary solution when $e(\mathbf{r}, t) = \bar{e}(r)$ and $\mathbf{v}(\mathbf{r}, t) = 0$ and also \bar{e} satisfies, [9]

$$-K(\nabla^2 \bar{e}) + \frac{\partial f}{\partial \bar{e}} = 0. \quad (13)$$

We can now write the equations of motion for small deviations about the stationary configuration by defining $e = \bar{e}(r) + \nu(\mathbf{r}, t)$ where ν is a small fluctuation in energy

density and $\mathbf{v} = 0 + \mathbf{v}(\mathbf{r}, t)$ and linearizing Eqs. (8) and (10) around this configuration

$$\partial_t \nu(\mathbf{r}, t) = -\nabla \cdot (\omega \mathbf{v}(\mathbf{r}, t)), \quad (14)$$

$$\begin{aligned} \partial_t(\bar{\omega} \mathbf{v}(\mathbf{r}, t)) = & \nabla \bar{e} (-K \nabla^2 + f'') \nu(\mathbf{r}, t) \\ & + \nabla \cdot \left(\left(\frac{4}{3} \eta + \zeta \right) \nabla \cdot \mathbf{v}(\mathbf{r}, t) \right). \end{aligned} \quad (15)$$

Here $f'' = \partial^2 f / \partial e^2$, evaluated around the stationary configuration. The dynamical prefactor κ is determined with the radial perturbations of the form

$$\begin{aligned} \nu(\mathbf{r}, t) &= \nu(r) e^{\kappa t}, \\ \mathbf{v}(\mathbf{r}, t) &= \mathbf{v}(r) e^{\kappa t}. \end{aligned} \quad (16)$$

It can be seen from Eqs. (14) and (15) that the radial deviations are governed by the equations of motion of the following form

$$\begin{aligned} \kappa \nu(r) &= -\nabla \cdot (\bar{\omega} \mathbf{v}(r)), \\ \kappa \bar{\omega} \mathbf{v}(r) &= \nabla \bar{e} (-K \nabla^2 + f'') \nu(r) \\ &+ \nabla \cdot \left(\left(\frac{4}{3} \eta + \zeta \right) \nabla \cdot \mathbf{v}(r) \right). \end{aligned} \quad (17)$$

Following Langer and Turski, we find the solution for $\nu(r)$ in each of the three regions (i) the interior region of hadron phase, $r \leq R - \xi$ (ii) the exterior region of QGP phase, $r \geq R + \xi$ and (iii) the interface region, $R - \xi \leq r \leq R + \xi$, where R is the radius of the hadron bubble with origin at $r = 0$. The interfacial region has a thickness of the order of the correlation length ξ . Further, it is assumed that, everywhere outside the droplet, the energy density $\bar{e}(r)$ has the value e_q , the quark density. Within the droplet, $\bar{e}(r)$ is equal to the hadron density e_h . Thus, $\bar{e}(r)$ describes a smooth interfacial profile at $r = R$ going from e_h to e_q within a region of roughly of the order of the correlation length ξ . We then evaluate the relative amplitudes in the above three regions by matching the values at the boundaries. Finally, we evaluate κ applying the condition

$$\int_0^\infty r^2 \nu(r) dr = 0. \quad (18)$$

which is the conservation law implied by Eq. (14)

Before we proceed further, it may be noted here that the above set of linear equations [Eqs. (17)] is obtained from Eq. (10) which contains $f(e)$ and K explicitly. As will be shown subsequently, this form is suitable for the interfacial region where $\nabla \bar{e}$ is nonzero. Such an equation has been used in Ref. [9] to evaluate the velocity profile at the surface region in order to evaluate the energy loss due to dissipation. In another approach, Ruggeri-Friedman [11] linearize Eq. (9) only in the exterior region with the assumption $p = c_s^2 e$ where c_s is the velocity of sound in the medium. We will discuss about the validity

of the above assumption particularly near the stationary configuration as we proceed further. However, the advantage of using Eq.(9) along with the above relation is that one of the variable (say p) can be eliminated so that the linear equation becomes a simple wave equation. Therefore, we adopt both the approaches as in the following. We solve the relativistic hydrodynamic equations by linearising Eqs. (8) and (9) in the interior and exterior regions whereas we use the linear Eq. (17) in the interfacial region.

A. The interior and exterior region

In these regions, $\bar{e}(r)$ is varying so slowly that the gradient energy can be ignored so that Eq. (11) is consistent with

$$\nabla e \frac{\partial f}{\partial e} = \nabla f \rightarrow -\nabla p \quad (19)$$

Since $\partial f / \partial \bar{e} = 0$ at the saddle point [see Eq. (13)], the above relation would imply \bar{p} is independent of the energy density at the stationary configuration. Like energy density ν , we also consider a small fluctuation in pressure so that we have $p(r) = \bar{p} + \beta(r, t)$ (recall, $e(r) = \bar{e}(r) + \nu(r, t)$). We assume that the corresponding fluctuations satisfy the relation $\beta = c_s^2 \nu$ where c_s^2 is a constant (c_s could be the velocity of sound in the medium around the saddle configuration). Therefore, in the interior and exterior regions, we solve the equation

$$\kappa^2 \nu(r) = c_s^2 \nabla^2 \nu(r) + \frac{\kappa}{\bar{\omega}} \left(\frac{4}{3} \eta + \zeta \right) \nabla^2 \nu(r), \quad (20)$$

obtained from Eqs. (8) and (9) after linearizing around the stationary configuration and also using the relation $\nabla^2 \beta = c_s^2 \nabla^2 \nu$. Such a relation has also been used in ref [11], but we differ in our interpretation of the pressure gradient.

Assuming spherically symmetric solutions of the form

$$\nu(r) = \frac{\text{Constant}}{r} e^{\pm qr}, \quad (21)$$

we get the relation

$$\bar{\kappa}^2 = c_s^2 q^2, \quad (22)$$

where

$$\bar{\kappa}^2 = \frac{\kappa^2}{\left(1 + \frac{\kappa}{\bar{\omega} c_s^2} \left(\frac{4}{3} \eta + \zeta \right) \right)}. \quad (23)$$

The above relation holds both for QGP and the hadron regions, except for the fact that the viscosity coefficients are different in two phases. The interior and exterior solutions, therefore, are

$$\nu(r) = \frac{A}{r} \sinh(qr) \quad \text{for } 0 \leq r \leq R - \xi \quad (24)$$

and

$$\nu(r) = \frac{B}{r} e^{-q(r-R)} \quad \text{for } r \geq R + \xi. \quad (25)$$

If κ is small, the solution will be the one in which $\nu(r)$ varies slowly over a distance of the order of correlation length ξ so that $q\xi \ll 1$. Since κ is related to q , next we proceed to estimate it by solving the linear hydrodynamic equation in the interfacial region and matching it at the boundary.

B. Interfacial region

As will be shown subsequently, the velocity varies as $v \propto r^{-2}$ in this region so that $r^2 v$ remains constant. As a consequence, $\nabla \cdot \mathbf{v} = r^{-2} d(r^2 v)/dr$ vanishes at the surface region. Therefore, ignoring the viscous term and eliminating \mathbf{v} from Eqs. (17), an equation for $\nu(r)$ is obtained as

$$\kappa^2 \nu(r) = -\nabla \cdot [\nabla \bar{e} (-K \nabla^2 + f'') \nu(r)]. \quad (26)$$

Further, κ is assumed to be small so that in the first approximation, we can completely neglect the terms containing κ^2 . Thus, to a good approximation in the interfacial region, $\nu(r)$ satisfies

$$\nabla \cdot \left[\frac{d\bar{e}}{dr} (-K \nabla^2 + f'') \nu(r) \right] = 0. \quad (27)$$

We follow the procedure described in Ref. [6] to get the solution of the above equation in the interfacial region (see appendix A for detail),

$$\nu(r) \approx -\frac{a(R) R^2 \Delta \bar{e}}{2\sigma r} \frac{d\bar{e}}{dr}, \quad (28)$$

where $\Delta \bar{e} = e_q - e_h$. This solution is quite similar to that found in Ref. [6], with \bar{n} replaced by \bar{e} and a which is now a function of r evaluated at R . Note that in the quark and the hadron regions (far away from the surface region), $\bar{e}(r)$ is nearly constant. Thus, Eq. (27) becomes undefined in these regions. Therefore, a different set of equations has been used in the exterior-interior regions as discussed in the previous sections.

We can also get an expression for velocity $v(r)$ from the relation,

$$\kappa \nu(r) = \frac{1}{r^2} \frac{d}{dr} [r^2 \bar{\omega} v(r)], \quad (29)$$

Substituting $\nu(r)$ from Eq. (28), we get

$$v(r) = \frac{D}{r^2 \bar{\omega}} \int_0^r r dr \frac{d\bar{e}}{dr}, \quad (30)$$

where D is a constant. The above equation can be integrated to give

$$v(r) \approx \frac{D}{\bar{\omega}_q} \frac{R}{r^2}. \quad (31)$$

Recall that this result is consistent with our assumption that $r^2 v$ is constant in the surface region.

C. Dynamical Prefactor

For the interfacial region, the solution is given in the appendix A. It remains now only to apply Eq. (18) to compute q (or κ). As in [6], we can neglect the contribution coming from the interior region ($r < R$) and the terms of order $qR \approx \sqrt{\xi/R}$ in the exterior region. The contribution coming from the interfacial region is $\approx -aR^3(\Delta\bar{e})^2/2\sigma$ where a is a function of r related to the constant B and the second derivative of f w.r.t. \bar{e}_q . The exterior region contribution is $\approx B/q^2$. Combining both the terms, we get

$$q = \sqrt{\frac{2\sigma B}{a(R)R^3(\Delta\bar{e})^2}}. \quad (32)$$

Assuming $a(R + \xi) = a(R - \xi) \approx a(R)$ and using the relation $(\partial^2 f / \partial \bar{e}_q^2)^{-1}$ for B/a [see Eq. (A7)], we obtain

$$q = \sqrt{\frac{2\sigma}{R^3(\Delta\bar{e})^2} \frac{1}{(\partial^2 f / \partial \bar{e}_q^2)}}. \quad (33)$$

We can also eliminate $\partial^2 f / \partial \bar{e}_q^2$ by using the relation

$$\frac{1}{K} \frac{\partial^2 f}{\partial \bar{e}_q^2} = \frac{1}{\xi_q^2}, \quad (34)$$

where ξ_q is the correlation length and K is related to the surface tension σ given by Eq. (12). The choice of K depends on the energy density profile $\bar{e}(r)$. Following [9], σ can be related to K in the planar interface approximation at T_c as

$$\sigma = K(\Delta\bar{e})^2/6\xi_q, \quad (35)$$

which will result in

$$q = \sqrt{\frac{\xi_q}{3R^3}}. \quad (36)$$

Therefore, in the case of a non-viscous plasma, we get a very simple relation for κ given by

$$\kappa = c_s \sqrt{\frac{\xi_q}{3R^3}} = \xi_q^{-1} \sqrt{\frac{c_s^2 x^3}{3}} = \xi_q^{-1} f(x), \quad (37)$$

where $x = \xi_q/R$. This can be viewed as the critical behavior of κ that scales as $\xi_q^{-1} f(x)$. However, this scaling law is different from the dynamical scaling law that one finds in the case of a non-relativistic liquid-vapor transition [7,8] where κ scales as $\xi^0 R^{-3}$. While this needs further investigation, one of the reason for this discrepancy could be unlike the static scaling, the dynamical scaling depends on the dynamical behavior of the system [16] which is definitely different depending on whether the medium is relativistic or non-relativistic. The above result is also valid in the case of the viscous plasma, only instead of κ , $\bar{\kappa}$ will scale as $\xi_q^{-1} f(x)$. Therefore,

for viscous quark gluon plasma, this scaling results in a quadratic equation in κ with solution given by

$$\kappa = \frac{\alpha q^2}{2} + c_s q \sqrt{1 + \frac{\alpha^2 q^2}{4 c_s^2}}, \quad (38)$$

where $\alpha = (4\eta/3 + \zeta)/\bar{\omega}$. Since in the first approximation $q \propto \kappa$, we can neglect the second term under the square root which are higher order in q^2 and α^2 (viscosity). Finally, we get

$$\kappa = \frac{q^2}{2\bar{\omega}} \left(\frac{4}{3}\eta + \xi \right) + c_s q. \quad (39)$$

Using Eq. (36) for q , we can obtain a general expression for κ for a viscous QGP as

$$\kappa = c_s \sqrt{\frac{\xi_q}{3R^3}} + \frac{\xi_q}{6R^3} \frac{1}{\bar{\omega}_q} \left(\frac{4}{3}\eta_q + \zeta_q \right). \quad (40)$$

Therefore, the prefactor κ can be written as the sum of two terms having a non-viscous (κ_0) and a viscous (κ_v) component. However, both κ_0 and κ_v have simple dependence on the correlation length ξ_q and the bubble radius R . As can be seen from Eq. (40), the first term is more dominating as compared to the second one particularly when T is close to T_c . However, as temperature decreases, the viscous contribution competes with that of the non-viscous one.

We can also express the above equation (40) in a different way by assuming c_s^2 as

$$c_s^2 = \bar{\omega}_q \frac{\partial^2 f}{\partial \bar{e}_q^2}, \quad (41)$$

The above relation is analogous to the non-relativistic expression for velocity of sound in the medium which has a similar relation with ω and e replaced by n (density) [11]. Then from Eq. (33) we get

$$q = c_s^{-1} \sqrt{\frac{2\sigma}{R^3} \frac{\bar{\omega}_q}{(\Delta\bar{\omega})^2}}, \quad (42)$$

where we have used the approximation $\Delta\bar{\omega} \approx \Delta\bar{e}$ since the pressure difference is negligible as compared to the difference in energy density. Now using the above q in Eq. (39), the prefactor κ can be written as

$$\kappa = \sqrt{\frac{2\sigma}{R^3} \frac{\bar{\omega}_q}{(\Delta\bar{\omega})^2}} + \frac{1}{c_s^2} \frac{\sigma}{R^3(\Delta\bar{\omega})^2} \left(\frac{4}{3}\eta_q + \zeta_q \right). \quad (43)$$

As can be seen, the first term in the above equation is same as Eq. (4) as obtained by Ruggeri and Friedman corresponding to the case of a non-viscous plasma. The second term is similar to the result obtained by Csernai and Kapusta except with a minor difference, i.e., instead of 4, we have a factor of c_s^{-2} in the numerator [see Eq. (3)].

However this is a small difference which can be removed by redefining K [see Eq. (35)].

It may be mentioned here that the relation $\beta = c_s^2 \nu$ has been used to obtain κ as given by Eq. (40). This is the main result of our work. It is also satisfying to note that we can recover the result of Csernai-Kapusta and Ruggeri-Friedman under the assumption for c_s^2 given by Eq. (41). In analogy with the non-relativistic case, we may interpret c_s as the velocity of sound around the saddle configuration (Recall that saddle point is the configuration where Eq. (13) is satisfied). Although, the above interpretation needs further justification, it is sufficient to say that the results of Eq. (43) can be recovered using c_s^2 as given by Eq. (41).

D. Result and discussion

We should point out here that there are several reasonable assumptions that have been made in our derivation of the dynamical prefactor. As discussed in the text, most of these approximations are same as that of used in the original work of Langer-Turski [6] since we use the same procedure except that the equations follow relativistic hydrodynamics. An important aspect where we differ from both Ref [9] and Ref [6] is the use of the relation $\beta = c_s^2 \nu$ assumed to be valid in the quark and hadron regions, (β and ν are the radial deviations of the pressure and energy density from the stationary solution). Although, we differ in our interpretation, such a relation has also been used by Ruggeri and Friedman [11] to eliminate one of the hydrodynamic variables. However, we do not make any such assumption in the interfacial region. As a result, the linearized equation used in the interfacial region is different from the one used in the exterior-interior regions. Within above formalism, we have derived an expression for the dynamical prefactor κ . Two important aspects of our result are (a) the prefactor κ can be written as a linear sum of a non-viscous (κ_0) and a viscous (κ_v) component and (b) the non-viscous component (κ_0) which depends on two parameters R and ξ is finite in the limit of zero viscosity. The present result on κ_0 is also in agreement with the view point of Ruggeri-Friedman that the viscosity is not essential for the dynamical growth of the hadron bubbles. This fact is also evident from Eq. (8) which implies a non-vanishing energy flow $\omega \mathbf{v}$ even in the absence of viscosity. Only terms second order in \mathbf{v} appear in the energy equation in the presence of viscosity and the momentum equation contains a term linear in \mathbf{v} . This means that viscosity terms are relatively unimportant in the energy transport for small value of \mathbf{v} . The momentum equation, however, indicates that viscosity influences the time evolution of \mathbf{v} . Thus, viscosity can serve to disrupt the energy flow and generate entropy but cannot be the only mechanism for energy removal.

The above aspect apparently is in contradiction to the

general expectation that transport coefficients like viscosity and thermal conductivity are essential for the removal of the latent heat and hence for the growth of the hadron bubbles [7–10]. This point needs further clarification. The formation of the hadron bubbles can be interpreted as the thermal fluctuation of the new phase within a correlated volume of radius R and surface thickness ξ . According to the fluctuation-dissipation theory, this fluctuation would mean certain amount of heat dissipation. We can understand the origin of this heat dissipation as follows. In the absence of thermal conductivity, the dissipative losses that occur in a fluid are due to the coefficients of shear and bulk viscosity that depend on the gradient and divergence of the velocity field respectively. In case of an incompressible fluid, the losses are due to the shear stress alone since the bulk viscosity that provides resistance to the expansion (or contraction) does not exist. However, due to the nucleation of the critical size hadron bubble, the pressure or the tension in the fluid is no longer uniform, the pressure inside the hadron bubble being more than the outside. Due to this pressure difference, the hadron bubbles will keep expanding with a non-zero wall velocity. Thus, the fluid medium outside will exert a frictional force on the bubble wall (causing heat dissipation) whose magnitude depends on the pressure difference between the two phases [17,18]. In the field theoretical language, this dissipation corresponds to the coupling of the order parameter ϕ to the fluid which acts as heat bath. Estimates for it in the context of electroweak theory have been given in Refs. [19,20]. Therefore, our non-viscous part of the prefactor corresponds to a dissipation of dynamical nature which does not depend on any transport coefficients like viscosity or thermal conductivity. This dissipation basically arises due to non-uniform pressure across the interface. Following a different approach, Ignatius [21] had also derived κ in the limit of zero fluid velocity to be $\approx 2/(\eta R_c^2)$ where η is a phenomenological friction parameter (not to be confused with the shear viscosity of the plasma) responsible for the energy transportation between the order parameter and the fluid. Recently, Alamoudi et al [22] have also studied the dynamical viscosity and the growth rate of the nucleating bubble where the viscosity effects arise due to the interaction of the unstable co-ordinate with the stable fluctuations. They estimate a growth rate which depends on R , ξ and the self coupling λ ($\kappa \approx \frac{\sqrt{2}}{R} [1 - .003 \lambda T \xi (\frac{R}{\xi})^2]$). In the limit of weak coupling, the above growth rate scales as R^{-1} which is also consistent with our result [23].

Finally, we conclude this section with the comment that in case of relativistic heavy ion collisions, appreciable amount of nucleation begins from a super cooled metastable QGP phase at which the radius of the critical hadron bubble is of the same order as the width of the bubble interface. At such point, the homogeneous nucleation theory may break down. However, the system moves out of this problematic region quickly due to

the release of latent heat which heats up the medium again towards T_c . In the other application, in case of cosmology, this problem is not serious, although there the actual value of the dynamical growth rate may be of less important. In either case, this study has significance for homogeneous nucleation under the thin-walled bubble approximation.

IV. NUMERICAL RESULTS

In the following, we compare κ obtained from different methods. In the case of a second order phase transition, the correlation length ξ scales in the proximity of the critical point as $\xi(T) = \xi(0)(1 - T_0/T)^{-\nu}$ where $\nu = 0.63$ [24]. However, in the case of a first order phase transition, the transition temperature T_0 is smaller than T_c and approaches T_c only in the limit when strength of the transition becomes weak. Therefore, unlike the second order case, ξ_q at T_c will be finite and which, in the present context, represents the thickness of the interfacial region such that $R \gg \xi_q$. Further, we ignore the temperature dependence of σ and ξ_q and treat them as constant parameters. This assumption can be justified when the amount of supercooling is small and the medium returns to T_c due to the release of latent heat [2].

Figure 1 shows the temperature dependence of κ given by Eq. (40) along with viscous (κ_v) and non-viscous (κ_0) components at two different values of σ . Following [3], we take η_q as $2.5 T^3$ and set ζ_q to zero. With decreasing temperature as well as with decreasing σ , the value of the critical radius [which is obtained from Laplace formulae, see Eq. (46)] decreases. Therefore, the κ , κ_0 and κ_v increase with decreasing temperature and also they have higher values for smaller σ , as expected. The behavior of κ_v is quite different from that of κ_0 . Initially, near $T \approx T_c$, the κ_v has small value, but it exceeds κ_0 as temperature comes down particularly at smaller σ values. Figure 2 shows the similar plot as that of Figure 1 where we have used Eq. (43) to estimate κ . As seen from the figures, both the estimates have similar behavior although Eq.(43) yields slightly higher values for κ as compared to Eq.(40). The above studies also suggest that the effect of viscosity is negligible at higher σ values and also for small amount of supercooling. However, its effect can not be ignored at much lower temperature particularly when σ is small. In figure 3, we have also compared only the non-viscous part (κ_0) of the prefactor as obtained from Eqs.(40) and (43) at two different values of ξ_q . Within the present set of parameters, the non-viscous parts of the prefactor obtained by both the methods behave similar way.

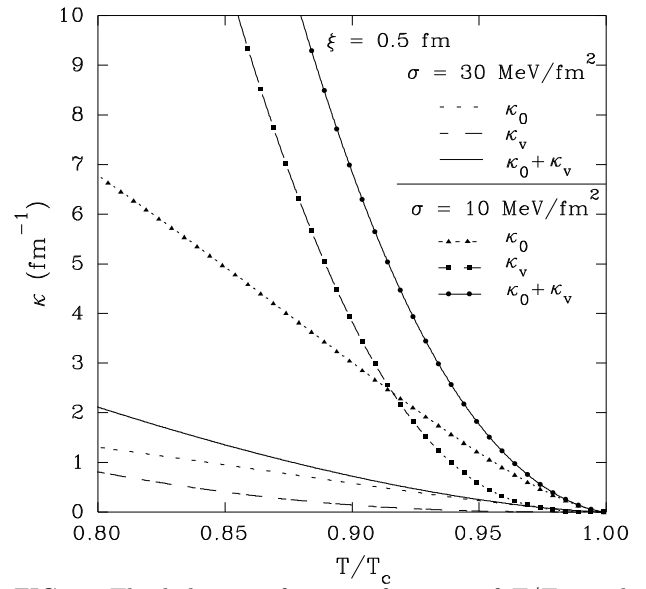


FIG. 1. The behavior of κ as a function of T/T_c as obtained from Eq. (40). Short-dashed curve is for non-viscous component κ_0 , long-dashed curve is viscous component κ_v and solid curve is for the sum $\kappa_0 + \kappa_v$ with $\sigma = 30 \text{ MeV/fm}^2$. The corresponding curves at $\sigma = 10 \text{ MeV/fm}^2$ are shown by triangles, squares and circles respectively.

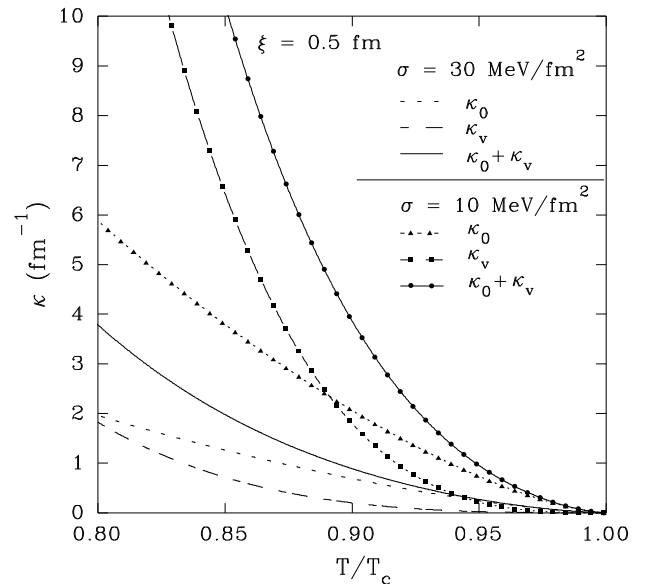


FIG. 2. The behavior of κ as a function of T/T_c as obtained from Eq. (43). Lines and symbols are same as in Fig. 1

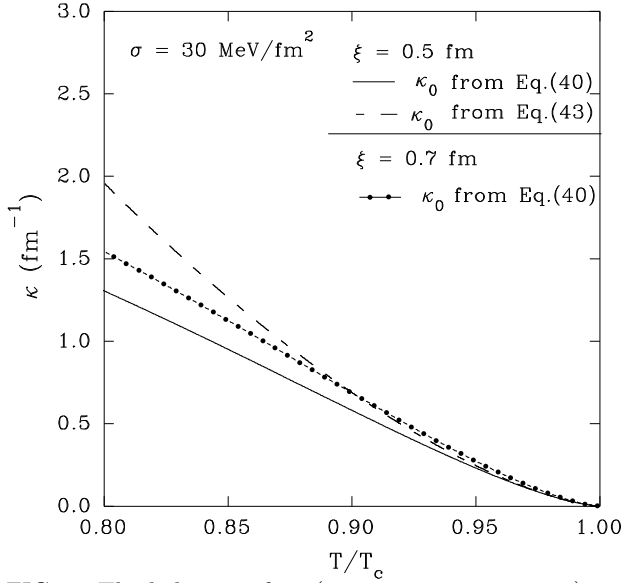


FIG. 3. The behavior of κ_0 (non-viscous component) as a function of T/T_c . The solid curves are from Eq. (40) while the dashed curve is from Eq. (43) shown at two typical values of ξ_q .

We also study the dynamics of the nucleation and super cooling by computing the nucleation rate as

$$I = \frac{\kappa}{2\pi} \frac{\Omega_0}{V} e^{-F_C/T}, \quad (44)$$

where F_C is free energy needed to form a critical bubble in the metastable (supercooled) background. The dynamical prefactor κ is estimated using Eq. (40) whereas the statistical prefactor Ω_0 is taken from the previous works [9,2] as

$$\frac{\Omega_0}{V} = \frac{2}{3} \left(\frac{\sigma}{3T} \right)^{3/2} \left(\frac{R}{\xi_q} \right)^4, \quad (45)$$

where R is the radius of the critical bubble. Under thin-wall approximation F_C and R for a spherical bubble are given by

$$F_C = \frac{4\pi}{3} \sigma R^2, \quad R = \frac{2\sigma}{p_h - p_q}. \quad (46)$$

From the nucleation rate $I(T)$, the fraction of space which has been converted to hadron phase can be calculated. If the system cools to T_c at a proper time τ_c , then at some later time τ the fraction h of space which has been converted to hadronic gas [2] is

$$h(\tau) = \int_{\tau_c}^{\tau} d\tau' I(T(\tau')) [1 - h(\tau')] V(\tau', \tau). \quad (47)$$

Here $V(\tau', \tau)$ is the volume of a bubble at time τ which had been nucleated at an earlier time τ' ; this takes into account the bubble growth. The factor $[1 - h(\tau')]$ is the available space for new bubbles to nucleate. The model for bubble growth is simply taken as [25]

$$V(\tau', \tau) = \frac{4\pi}{3} \left(R(T(\tau')) + \int_{\tau'}^{\tau} d\tau'' v(T(\tau'')) \right)^3, \quad (48)$$

where $v(T) = 3[1 - T/T_c]^{3/2}$ is the velocity of the bubble growth at temperature T [2]. (Recall that this velocity is different from the velocity of the nucleated bubble surface as used in the previous sections). The evolution of the energy momentum in 1+1 dimension is given by

$$\frac{de}{d\tau} + \frac{\omega}{\tau} = \frac{\frac{4}{3}\eta + \zeta}{\tau^2}. \quad (49)$$

In this work, we use the bag equation of state for QGP. The energy and enthalpy densities in pure QGP and hadron phases are taken as

$$e_q(T) = 3a_q T^4 + B, \quad \omega_q(T) = 4a_q T^4, \quad (50)$$

$$e_h(T) = 3a_h T^4, \quad \omega_h(T) = 4a_h T^4. \quad (51)$$

Here, a_q and a_h are related to the degrees of freedom operating in two phases and B is the bag pressure. The quark phase is assumed to consist of massless gas of u , d quarks and gluon while the hadron phase contains massless pions. Thus the coefficients $a_q = 37\pi^2/90$ and $a_h = 3\pi^2/90$. In the transition region, the energy density at a time τ can be written in terms of hadronic fraction $h(\tau)$ as

$$e(\tau) = e_q(T) + (e_h(T) - e_q(T))h(\tau). \quad (52)$$

Following [15,3], the viscosity coefficients for the QGP and the hadron phases are chosen as $\eta_q = 2.5T^3$, $\zeta_q = 0$, $\eta_h = 1.5T^3$ and $\zeta_h = T^3$. The other parameters are $T_c = 160$ MeV and $\sigma = 30$ MeV/fm³. With the above set of parameters, Eqs. (49) and (47) are solved to get h and T as a function of time τ [2,3].

Figures 4(a) and 4(b) show the plot of $s\tau$ - the rate of entropy production and T/T_c - the rate of supercooling as a function of τ both for ideal hydrodynamic (IHD) and viscous hydrodynamic (VHD) expansions of the system. The system cools below T_c until nucleation rate becomes significant. Afterwards, bubble nucleation and growth reheats the system due to the release of latent heat. This behavior is similar to what has been studied earlier in Refs. [2,3] using a prefactor which explicitly depends on the viscosity coefficient of the plasma. In the present work, since κ has both viscous and non-viscous components, we study the supercooling and extra entropy production both with κ_0 and κ particularly when the medium is non-viscous. First we consider only the ideal hydrodynamic expansion. The short-dashed curve and the long-dashed curves are obtained using Eq. (40) for κ_0 (no viscosity) and κ (κ_v included) respectively. Since there is supercooling with κ_0 , extra entropy is generated even without viscosity. As shown earlier (see Figures 1 and 2), the effect of viscosity on κ is not significant with a reasonable choice of $\eta_q = 2.5T^3$ particularly for small

amount of supercooling. Therefore, inclusion of κ_v does not effect the supercooling much (see the long-dashed curve). The supercooling (hence the entropy production) comes down only by about $\approx 1\%$ due to viscosity, with the present set of parameters. As mentioned before, even-though we use κ , we do not include viscosity in the hydrodynamical evolution just to bring out the additional effect due to the use of κ instead of κ_0 in the prefactor. However, when the plasma is viscous, the VHD should be used for consistency (i.e. when κ_v is included). The use of VHD reduces the supercooling by about 10 % as shown by the solid curve. Although, the amount of supercooling reduces, the entropy production goes up. Since the effect of viscosity on κ is insignificant, the reduction in supercooling is purely due to the viscous heating of the medium. As a result extra entropy is generated in addition to the entropy that is produced due to supercooling.

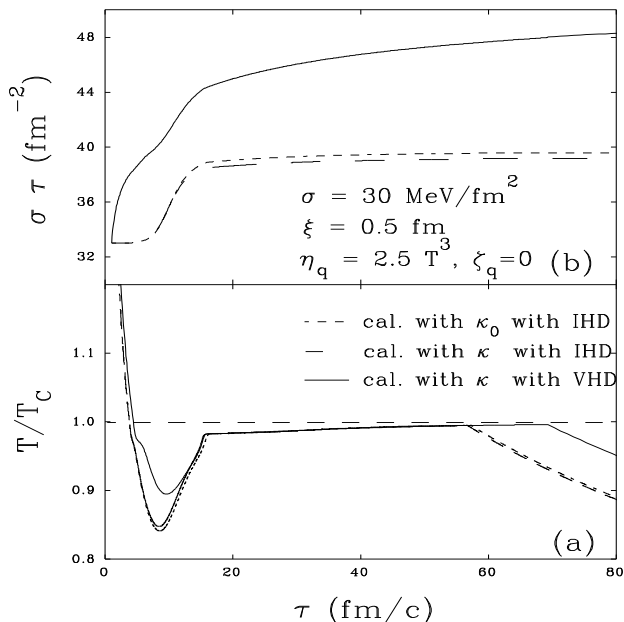


FIG. 4. (a) $s\tau$ - the rate of entropy production and (b) T/T_c - the rate of supercooling as a function of τ , short-dashed curves are calculated with κ_0 from Eq. (40) using Ideal Hydrodynamics (IHD), the long-dashed curves are calculated with κ using IHD, the solid curves are calculated with κ but using viscous Hydrodynamics (VHD).

V. CONCLUSION

To summarize, we have derived an expression for the dynamical prefactor which governs the initial growth of critical size bubbles nucleated in first order phase transition. In the case of a non-viscous plasma, the dynamical growth rate is found to depend only on the correlation length and the size of the hadron bubble which are two meaningful scale parameters to describe the critical phenomena at the transition point. The correction to the dynamical prefactor due to viscosity is found to be additive and does not affect the growth process significantly

though additional entropy is generated due to viscous heating of the medium. Since the prefactor does not vanish in the limit of zero viscosity, extra entropy is produced during the process of nucleation even when the fluid is non-viscous. Nearly similar conclusions are also drawn by Ruggeri and Friedman who had derived dynamical prefactor by solving relativistic hydrodynamics following a different approach. However, unlike their result, the present prefactor can be written as the sum of viscous and non-viscous terms. Interestingly, using an assumption for velocity of sound in the medium (around the saddle configuration) which has a form analogous to what is used for non-relativistic plasma, the viscous and the non-viscous parts are found to be similar to the results as obtained by Csernai-Kapusta and Ruggeri-Friedman respectively.

In the present work we solve relativistic hydrodynamic equations both in the interior-exterior, i.e., quark-hadron regions and surface regions. The linear hydrodynamic equation used in the quark-hadron region is obtained after eliminating one of the variable using the relation $\beta = c_s^2 \nu$ which is not valid in the surface region. Therefore, a different equation is used for the surface region which involves the extra gradient energy. This is where we differ from the Csernai and Kapusta method. Further, Csernai and Kapusta derived κ by equating the flow of the outward energy flux with the dissipative loss due to viscosity of the medium and the contribution due to a dynamical dissipation was not included. On the other hand, Ruggeri and Friedman solve the hydrodynamic equation only in the quark region and use a set of boundary conditions with certain assumptions. In this context, the present formalism is more general as we solve the linearized hydrodynamic equations in all space and obtain an expression for the prefactor by matching the solutions at the boundary of the interface. Moreover, our result is different in the sense that it has a very simple dependence on the correlation length and radius of the hadron bubble although the CK and RF results can be obtained from it under certain assumption.

APPENDIX: A

We find out the solution for $\nu(r)$ that satisfies the radial equation (see Eq. 27),

$$\frac{d}{dr} \left[r \frac{d\bar{\epsilon}}{dr} \left(-K \frac{d^2}{dr^2} + f'' \right) \chi(r) \right] = 0, \quad (\text{A1})$$

where $\nu(r) = \chi(r)/r$. Finally we solve for $\chi(r)$ from the equation

$$\left(-K \frac{d^2}{dr^2} + \frac{\partial^2 f}{\partial \bar{\epsilon}^2} \right) \chi(r) = a(r). \quad (\text{A2})$$

The above equation is quite identical to the one used by Langer-Turski [6] in the surface region. The only

difference is that the constant a now depends on r as $a(r) \propto (r\nabla\bar{e})^{-1}$. (Note that $\nabla\bar{e}$ peaks at $r \approx R$). Since $R \gg \xi$ and $\nabla\bar{e}(r)$ varies sharply in the range $R - \xi \leq r \leq R + \xi$, $a(r)$ mostly depends on the $\nabla\bar{e}(r)$ variation. Therefore, to a good approximation we can write $a(r) \propto (R\nabla\bar{e})^{-1}$. The general form of the solution of Eq. (A2) is given by

$$\chi(r) = \int dr' G(r, r') a(r'), \quad (\text{A3})$$

where G is the Green's function satisfying

$$\left(-K \frac{d^2}{dr^2} + \frac{\partial^2 f}{\partial \bar{e}^2}\right) G(r, r') = \delta(r - r'). \quad (\text{A4})$$

On either side of the interface $\partial^2 f / \partial \bar{e}^2$ is nearly constant. Using the relation $\nabla^2 \bar{e} = 0$, it is easy to verify that

$$\chi(r) \approx a(r) \left(\frac{\partial^2 f}{\partial \bar{e}^2}\right)^{-1} \quad (\text{A5})$$

is an approximate solution of Eq. (A2) at the interface boundary. Matching the solution in the interfacial region given by Eq. (A5) with the solution in the interior region given by Eq. (24) at $R - \xi$ and with the solution in the exterior region Eq. (25) at $R + \xi$, give the following conditions

$$A \sinh(q_h R) = a(R - \xi) \left(\frac{\partial^2 f}{\partial \bar{e}_h^2}\right)^{-1} \quad (\text{A6})$$

and

$$B = a(R + \xi) \left(\frac{\partial^2 f}{\partial \bar{e}_q^2}\right)^{-1}. \quad (\text{A7})$$

Here the condition $q\xi \ll 1$ has been used. To get a solution inside the interface, we follow the same procedure as that of Ref. [6], i.e., we use the spectral decomposition of G as

$$G(r, r') = \sum_n \frac{\chi_n(r) \chi_n(r')}{\bar{\lambda}_n}, \quad (\text{A8})$$

where $\bar{\lambda}_n$ are the s-wave eigenvalues and χ_n are the corresponding eigenfunctions. For value of r near R , the sum will be dominated by the first term. This is because $\bar{\lambda}_1 \simeq -\frac{2K}{R^2}$, vanishes as R becomes large. Since $\chi_1(r) \simeq (\frac{K}{\sigma})^{1/2} (d\bar{e}/dr)$ is sharply peaked at interface, using Eqs. (A3) and (A8) we get

$$\chi(r) \approx -\frac{a(R) R^2 \Delta \bar{e}}{2\sigma} \frac{d\bar{e}}{dr}, \quad (\text{A9})$$

where $\Delta \bar{e} = e_q - e_h$. we can also estimate the variation of a in the range $R - \xi \leq r \leq R + \xi$,

$$\frac{a(R + \xi)}{a(R)} \approx \frac{\nabla \bar{e}(R)}{\nabla \bar{e}(R + \xi)}. \quad (\text{A10})$$

Assuming ξ to be the half width of the full maxima, the above ratio could be $\approx \sqrt{2}$.

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